A Gradient Scheme for the Discretization of Richards Equation

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Abstract We propose a finite volume method on general meshes for the discretization of Richards equation, an elliptic - parabolic equation modeling groundwater flow. The diffusion term, which can be anisotropic and heterogeneous, is discretized in a gradient scheme framework, which can be applied to a wide range of unstructured possibly non-matching polyhedral meshes in arbitrary space dimension. More precisely, we implement the SUSHI scheme which is also locally conservative. As is needed for Richards equation, the time discretization is fully implicit. We obtain a convergence result based upon energy-type estimates and the application of the Fréchet-Kolmogorov compactness theorem. We implement the scheme and present the results of a number of numerical tests.

1 Richards equation

In this article, we study Richards equation using Kirchhoff transformation. Let Ω be a open bounded polygonal subset of $\mathbb{R}^d$ ($d = 1,2$ or 3) and let T be a positive real number; Richards equation in the space-time domain $Q_T = \Omega \times (0,T)$ is given by

$$\partial_t \Big(\phi(\mathbf{x})\theta(p) \Big) - \operatorname{div}\Big(k_r(\theta(p))\mathbf{K}(\mathbf{x})\nabla(p+z) \Big) = 0, \tag{1}$$

where $p(\mathbf{x},t)$ is pressure head. The function $\theta(p)$ is the water saturation, $\phi(\mathbf{x})$ is the porosity, $\mathbf{K}(\mathbf{x})$ is the absolute permeability tensor and the scalar function $k_r(\theta)$

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corresponds to the relative permeability, which depends on the water content. The space coordinates are defined by $\mathbf{x} = (x,z)$ in the case of space dimension 2 and $\mathbf{x} = (x,y,z)$ in the case of space dimension 3. Next we perform Kirchhoff's transformation. We set

$$F(s) := \int_0^s k_r(\theta(\tau))d\tau,$$

and suppose that the function F is invertible. Then we set $u = F(p)$ in Q_T and $c(u) = c(F(p)) = \theta(p)$. We remark that Kirchhoff's transformation leads to $\nabla u = k_r(\theta(p))\nabla p$. Thus, the equation (1) becomes

$$\partial_t\Big(\phi(\mathbf{x})c(u)\Big) - \operatorname{div}\Big(\mathbf{K}(\mathbf{x})\nabla u\Big) - \operatorname{div}\Big(k_r(c(u))\mathbf{K}(\mathbf{x})\nabla z\Big) = 0. \tag{2}$$

Next, we consider the equation (2) together with the inhomogeneous Dirichlet boundary and the initial conditions

$$\begin{aligned} u(\mathbf{x},t) &= \hat{u}(\mathbf{x}) && \text{a.e. on } \partial\Omega \times (0,T), \\ u(\mathbf{x},0) &= u_0(\mathbf{x}) && \text{a.e. in } \Omega. \end{aligned} \tag{3}$$

We make the following hypotheses:
(H_1) c is a continuous nondecreasing function such that there $\overline{\xi} > 0$ and $\underline{\xi} \geq 0$ satisfying $|c(u)| \leq \overline{\xi}(1+|u|)$ for all $u \in \mathbb{R}$ and $|c(u)-c(v)| \geq \underline{\xi}|u-v|$ for all $u,v \in \mathbb{R}$.
(H_2) k_r is a continuous function such that $0 \leq k_r \leq \overline{k_r}$.
(H_3) $\mathbf{K}$ is a bounded function from Ω to $\mathbb{M}_d(\mathbb{R})$, where $\mathbb{M}_d(\mathbb{R})$ denotes the set of real $d \times d$ matrices. Moreover for a.e. $\mathbf{x}$ in Ω, $\mathbf{K}(\mathbf{x})$ is a symmetric positive definite matrix and there exist two positive constants $\overline{\mathrm{K}}$ and $\underline{\mathrm{K}}$ such that the eigenvalues of $\mathbf{K}(\mathbf{x})$ are included in $[\overline{\mathrm{K}}, \underline{\mathrm{K}}]$.
(H_4) $u_0 \in L^2(\Omega)$, $\hat{u} \in H^1(\Omega)$ and $\phi \in L^\infty(\Omega)$ is such that $0 < \underline{\phi} \leq \phi(\mathbf{x}) \leq \overline{\phi}$ for a.e. $\mathbf{x} \in \Omega$.

Definition. A function $u(\mathbf{x},t)$ is said to be a weak solution of Problem (2) - (3) if:

$$\begin{aligned} &(i) \quad u(\mathbf{x},t) - \hat{u}(\mathbf{x}) \in L^2(0,T;H_0^1(\Omega)), \\ &(ii) \quad c(u) \in L^\infty(0,T;L^2(\Omega)), \\ &(iii) \ -\int_0^T \int_\Omega \phi(\mathbf{x})c(u(\mathbf{x},t))\partial_t\varphi(\mathbf{x},t)\, d\mathbf{x}dt - \int_\Omega \phi(\mathbf{x})c(u_0(\mathbf{x}))\varphi(\mathbf{x},0)\, d\mathbf{x} \\ &\qquad\qquad + \int_0^T \int_\Omega \mathbf{K}(\mathbf{x})\nabla u(\mathbf{x},t) \cdot \nabla\varphi(\mathbf{x},t)\, d\mathbf{x}dt \\ &\qquad\qquad + \int_0^T \int_\Omega k_r(c(u(\mathbf{x},t))\mathbf{K}(\mathbf{x})\nabla z \cdot \nabla\varphi(\mathbf{x},t)\, d\mathbf{x}dt = 0, \end{aligned} \tag{4}$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega))$ with $\varphi(\cdot,T) = 0$ and $\partial_t\varphi \in L^\infty(Q_T)$.

The discretization of Richards equation by means of gradient schemes has already been proposed by Eymard, Guichard, Herbin and Masson [3], where they consider Richards equation as a special case of two phase flow; however, they make the extra hypothesis that the relative permeability k_r is bounded away from zero.

2 Gradient discretization

Following [2] we define a gradient discretization D of Problem (2) - (3) on a vector space X_D, or more precisely its subspace X_D^0 associated with the homogeneous Dirichlet boundary condition, and the two following linear operators:

- A gradient operator on the matrix domain: $\nabla_D : X_D \to L^2(\Omega)^d$.
- A function reconstruction operator on the matrix domain: $\pi_D : X_D \to L^2(\Omega)$.

Coercivity: We assume that $\|\nabla_D \cdot\|_{L^2(\Omega)^d}$ defines a norm on X_D^0. A gradient discretization D is said to be coercive if there exists $C_D \geq 0$ such that for all $v \in X_D^0$ one has

$$\|\pi_D v\|_{L^2(\Omega)} \leq C_D \|\nabla_D v\|_{L^2(\Omega)^d}.$$

Consistency: Let $u \in H_0^1(\Omega)$, and let us define

$$S_D(u) = \inf_{v \in X_D^0} \Big(\|\nabla_D v - \nabla u\|_{L^2(\Omega)^d} + \|\pi_D v - u\|_{L^2(\Omega)} \Big).$$

Then, a sequence of gradient discretizations $(D^{(m)})_{m\in\mathbb{N}}$ is said to be consistent if for all $u \in H_0^1(\Omega)$, $\lim_{m\to+\infty} S_{D^{(m)}}(u) = 0$.

Limit Conformity: For all $\mathbf{q} \in H_{div}(\Omega)$, we define

$$W_D(\mathbf{q}) = \sup_{0 \neq v \in X_D^0} \frac{1}{\|\nabla_D v\|_{L^2(\Omega)^d}} \int_\Omega \nabla_D v \cdot \mathbf{q} + \pi_D v \mathrm{div}(\mathbf{q})\, d\mathbf{x}. \tag{5}$$

Then, a sequence of gradient discretizations $(D^{(m)})_{m\in\mathbb{N}}$ is said to be limit conforming if for all $\mathbf{q} \in H_{div}(\Omega)$, $\lim_{m\to+\infty} W_{D^{(m)}}(\mathbf{q}) = 0$.

Compactness: A sequence of gradient discretizations $(D^{(m)})_{m\in\mathbb{N}}$ is said to be compact if for all sequences $v_m \in X^0_{D^{(m)}}$, $m \in \mathbb{N}$ such that there exists $C > 0$ with $\|\nabla_{D^{(m)}} v_m\|_{L^2(\Omega)^d} \leq C$ for all $m \in \mathbb{N}$, then there exist $\overline{v} \in L^2(\Omega)$ such that

$$\lim_{m\to+\infty} \|\pi_{D^{(m)}} v_m - \overline{v}\|_{L^2(\Omega)} = 0.$$

For $N \in \mathbb{N}^*$, let us consider the time discretization $t^0 = 0 < t^1 < \cdots < t^{n-1} < t^n \cdots < t^N = T$ of the time interval $[0,T]$. We denote the time steps by $\delta t^n = t^n - t^{n-1}$ for all $n \in \{1,\cdots,N\}$ while δt stands for the whole sequence $(\delta t^n)_{n\in\{1,\ldots,N\}}$. For all $v = \big(v^n \in X_D\big)_{n=1,\cdots,N}$ we set $\pi_{D,\delta t} v(\mathbf{x},t) = \pi_D v^n(\mathbf{x})$ and $\nabla_{D,\delta t} v(\mathbf{x},t) = \nabla_D v^n(\mathbf{x})$ for all $(\mathbf{x},t) \in \Omega \times (t^{n-1}, t^n]$, $n \in \{1,\ldots,N\}$.

Discrete variational formulation: For a given $u^0, \hat{u}_D \in X_D$ find $u = \big(u^n \in X_D\big)_{n\in\{1,\ldots,N\}}$ such that for each $n \in \{1,\ldots,N\}$, $u^n - \hat{u}_D \in X_D^0$ and for all $v \in X_D^0$

$$\int_{\Omega} \phi \frac{c(\pi_D u^n) - c(\pi_D u^{n-1})}{\delta t^n} \pi_D v \, d\mathbf{x} + \int_{\Omega} \mathbf{K}(\nabla_D u^n + k_r(\pi_D u^n)\nabla z) \cdot \nabla_D v \, d\mathbf{x} = 0 \quad (6)$$

Proposition 1 *There exists at least one solution of* (6)*; moreover there exists a positive C only depending on $\underline{\phi}, \overline{\phi}, \underline{\xi}, \overline{\xi}, \underline{K}, \overline{K}, \overline{k_r}$, Ω, T, u_0, $\hat{u}$ as well as on $\|c(\pi_D u^0) - c(u_0)\|_{L^2(\Omega)}$, $\|\pi_D \hat{u}_D - \hat{u}\|_{L^2(\Omega)}$ and $\|\nabla_D \hat{u}_D - \nabla \hat{u}\|_{L^2(\Omega)}$ such that*

$$\|c(\pi_{D,\delta t} u)\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla_{D,\delta t} u\|_{L^2(Q_T)^d} \leq C \quad (7)$$

for any solution u of (6).

Proof. In order to keep this presentation short, we only prove below the priori estimate (7), and only in the case of homogeneous Dirichlet boundary conditions; the adaptation to the inhomogeneous case is straightforward, and the existence of a discrete solution can be deduced using a standard argument based upon the topological degree. Let $u = (u^n)_{n\in\{1,\ldots,N\}}$ be a solution of (6) and define

$$\begin{aligned} A^n_{D,\delta t}(v) &= \int_{\Omega} \phi \frac{c(\pi_D u^n) - c(\pi_D u^{n-1})}{\delta t^n} \pi_D v \, d\mathbf{x}, \\ B^n_{D,\delta t}(v) &= \int_{\Omega} \mathbf{K}\nabla_D u^n \cdot \nabla_D v \, d\mathbf{x}, \quad C^n_{D,\delta t}(v) = \int_{\Omega} \mathbf{K} k_r(\pi_D u^n)\nabla z \cdot \nabla_D v \, d\mathbf{x}, \end{aligned} \quad (8)$$

for all $n \in \{1,\ldots,N\}$ and $v \in X^0_D$. The terms defined above satisfy

$$A^n_{D,\delta t}(v) + B^n_{D,\delta t}(v) + C^n_{D,\delta t}(v) = 0 \text{ for all } v \in X^0_D. \quad (9)$$

Let us first estimate $\sum_{n=1}^{m} \delta t^n A^n_{D,\delta t}(u^n)$ for $m \in \{1,\ldots,N\}$; we define

$$\xi(u) = c(u)u - \int_0^u c(\tau)\, d\tau \quad \text{ for all } u \in \mathbb{R}.$$

For all $a, b \in \mathbb{R}$, one has $\xi(a) - \xi(b) = (c(a) - c(b))a - \int_b^a (c(\tau) - c(b))\, d\tau$ and since c is nondecreasing we have that $\xi(a) - \xi(b) \leq (c(a) - c(b))a$. It implies that

$$\sum_{n=1}^{m} \delta t^n A^n_{D,\delta t}(u^n) \geq \int_{\Omega} \phi(\xi(\pi_D u^m) - \xi(\pi_D u^0))\, d\mathbf{x}. \quad (10)$$

For all $a \in \mathbb{R}$ it holds $\frac{1}{2}\underline{\xi} a^2 \leq \xi(u) \leq c(a)a \leq \frac{(c(a))^2}{\overline{\xi}}$, therefore

$$\sum_{n=1}^{m} \Delta t^n A^n_{D,\Delta t}(u^n) \geq \frac{1}{2} \underline{\xi\, \phi} \|\pi_D u^m\|^2_{L^2(\Omega)} - \frac{1}{\underline{\xi}\,\overline{\phi}} \|c(\pi_D u^0)\|^2_{L^2(\Omega)}. \quad (11)$$

Using the assumptions (H_2)-(H_3) we deduce that $B^n_{D,\delta t}(u^n) \geq \underline{K} \|\nabla_D u^n\|^2_{L^2(\Omega)^d}$ and that $C^n_{D,\delta t}(u^n) \leq \overline{k_r}\ \overline{K} |\Omega|^{1/2} \|\nabla_D u^n\|_{L^2(\Omega)^d}$ for all $n \in \{1,\ldots,N\}$. Combining these inequalities with (9) and (11) gives

$$\begin{aligned}&\tfrac{1}{2}\underline{\xi}\ \underline{\phi}\|\pi_D u^m\|^2_{L^2(\Omega)} + \underline{K}\sum_{n=1}^{m}\delta t^n\|\nabla_D u^n\|^2_{L^2(\Omega)^d}\\ &\qquad \le \frac{1}{\underline{\xi}\overline{\phi}}\|c(\pi_D u^0)\|^2_{L^2(\Omega)} + \overline{k_r}\ \overline{K}|\Omega|^{1/2}\sum_{n=1}^{m}\delta t^n\|\nabla_D u^n\|_{L^2(\Omega)^d}.\end{aligned}$$

Applying Young's inequality to the last term above, we obtain

$$\overline{k_r}\ \overline{K}|\Omega|^{1/2}\sum_{n=1}^{m}\delta t^n\|\nabla_D u^n\|_{L^2(\Omega)^d} \le \frac{1}{2\varepsilon}\overline{k_r}^2\overline{K}T|\Omega| + \frac{\varepsilon}{2}\overline{K}\sum_{n=1}^{m}\delta t^n\|\nabla_D u^n\|^2_{L^2(\Omega)^d}.$$

This leads to

$$\begin{aligned}&\frac{1}{2}\underline{\xi}\ \underline{\phi}\|(\pi_{D,\delta t}u)\|^2_{L^\infty(0,T;L^2(\Omega))} + (\underline{K} - \frac{\varepsilon}{2}\overline{K}|)\|\nabla_{D,\delta t}u\|^2_{L^2(Q_T)^d}\\ &\qquad \le \frac{1}{\underline{\xi}\overline{\phi}}\|c(\pi_D u^0)\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon}\overline{k_r}^2\overline{K}T|\Omega|.\end{aligned} \tag{12}$$

One completes the proof of the estimate (7) by choosing $\varepsilon = \underline{K}/\overline{K}$ and using the assumptions (H_1) and (H_4).

The following result is rather standard and given without proof.

Proposition 2 *Let u be a solution to* (6)*. There exists a positive constant C only depending on $\underline{\phi},\overline{\phi},\underline{\xi},\overline{\xi},\underline{K},\overline{K},\overline{k_r}$, Ω, T, u_0, $\hat{u}$ as well as on $\|c(\pi_D u^0) - c(u_0)\|_{L^2(\Omega)}$, $\|\pi_D\hat{u}_D - \hat{u}\|_{L^2(\Omega)}$ and $\|\nabla_D\hat{u}_D - \nabla\hat{u}\|_{L^2(\Omega)}$ such that for all $\tau\in(0,T)$, there holds*

$$\int_0^{T-\tau}\int_\Omega\Big(\pi_{D,\delta t}u(\mathbf{x},t+\tau) - \pi_{D,\delta t}u(\mathbf{x},t)\Big)^2 d\mathbf{x}dt \le C\tau.$$

Theorem 1. *Let $(D^{(m)},\delta t^{(m)})_{m\in\mathbb{N}}$ be a family of discretizations, where $(D^{(m)})_{m\in\mathbb{N}}$ assumed to be limit conforming, consistent, compact and uniformly coercive in the sense that there exist C_1 such that $C_{D^{(m)}} \le C_1$ for all $m\in\mathbb{N}$; moreover we assume that $\|c(\pi_{D^{(m)}}u^0_m) - c(u_0)\|_{L^2(\Omega)}$, $\|\pi_{D^{(m)}}\hat{u}_{D^{(m)}} - \hat{u}\|_{L^2(\Omega)}$ and $\|\nabla_{D^{(m)}}\hat{u}_{D^{(m)}} - \nabla\hat{u}\|_{L^2(\Omega)}$, $\max_n\delta t^{(m),n}$ tend to 0 as $m\to\infty$. Let u_m be a solution of* (6) *for all $m\in\mathbb{N}$. Then, up to a subsequence*

$$\begin{aligned}&\pi_{D^{(m)},\delta t^{(m)}}u_m \to \overline{u} \text{ in } L^2(Q_T),\\ &\nabla_{D^{(m)},\delta t^{(m)}}u_m \rightharpoonup \nabla\overline{u} \text{ in } L^2(Q_T)^d,\end{aligned}$$

where $\overline{u}\in L^2(0,T;H^1(\Omega))$ is a solution of (4)*.*

Proof. Using the compactness and the uniform coercivity of the sequence $D^{(m)}$ as well as Propositions 1 and 2, we deduce from Fréchet-Kolmogorov theorem that the sequence $\{\pi_{D^{(m)},\delta t^{(m)}}u_m - \pi_{D^{(m)}}\hat{u}_{D^{(m)}}\}$ is relatively compact in $L^2(Q_T)$. Therefore, we may extract a subsequence of $\{u_m\}$ (denoted again by $\{u_m\}$) such that $\pi_{D^{(m)},\delta t^{(m)}}u_m$ converges to some $\overline{u}\in L^2(Q_T)$ strongly in $L^2(Q_T)$ and $\nabla_{D^{(m)},\delta t^{(m)}}u_m$ is weakly convergent in $L^2(Q_T)$. It follows from Lemma 7.1 of [1] that the subsequence u_m can also be chosen in such way that $c(\pi_{D^{(m)},\delta t^{(m)}}u_m)$ and $k_r(c(\pi_{D^{(m)},\delta t^{(m)}}u_m))$

converge strongly in $L^2(Q_T)$ to $c(\overline{u})$ and $k_r(c(\overline{u}))$ respectively; moreover one deduces from (7) that $c(\overline{u}) \in L^\infty(0,T;L^2(\Omega))$. Finally we deduce from the limit conformity of the scheme that $\overline{u} - \hat{u} \in L^2(0,T;H_0^1(\Omega))$ and that $\nabla_{D^{(m)},\delta t^{(m)}} u_m \rightharpoonup \nabla\overline{u}$ in $L^2(Q_T)^d$ as $m \to +\infty$. Using again the limit conformity and consistency of the scheme we deduce that $\overline{u}$ is a weak solution of (4).

3 Numerical tests

3.1 The Hornung-Messing problem

The Hornung-Messing problem is a standard test (cf. for instance [5]). We consider a horizontal flow in a homogeneous ground $\Omega = [0,1]^2$ and set $T = 1$. The problem after Kirchhoff's transformation is given by Problem (2) with

$$c(u) = \theta(p) = \begin{cases} \pi^2/2 - 2arctan^2(\dfrac{u}{2-u}) & \text{if } p < 0, \\ \pi^2/2 & \text{otherwise,} \end{cases}$$

and suitable boundary and initial conditions. Let $s = x - z - t$, its solution is given:

$$u(x,z,t) = \begin{cases} \dfrac{2p(x,z,t)}{1+p(x,z,t)} & \text{if } p < 0, \\ 2p(x,z,t) & \text{otherwise,} \end{cases} \quad p(x,z,t) = \begin{cases} -s/2 & \text{if } s < 0, \\ -tan\Big(\dfrac{e^s-1}{e^s+1}\Big) & \text{otherwise.} \end{cases} \tag{13}$$

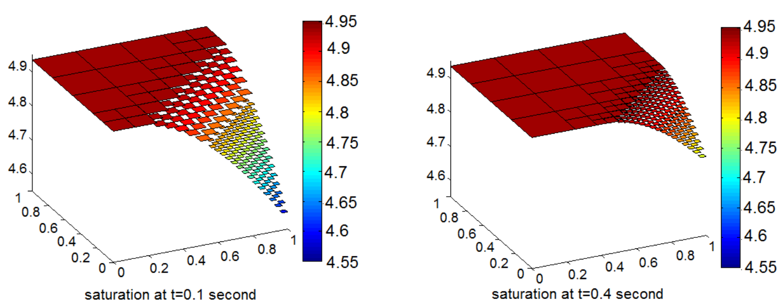

Fig. 1 Saturation at $t = 0.1$ seconds and at $t = 0.4$ seconds. The medium is unsaturated on the right-hand side of the space domain where $\theta < 4.9348$ and fully saturated elsewhere.

In this test, we apply the Sushi scheme [4] using an adaptive mesh driven by the variations of the saturation. We prescribe the Neumann boundary condition deduced from (13) on the line $x = 0$ and an inhomogeneous Dirichlet boundary condition elsewhere. We use an initially square mesh, which is such that each square can be decomposed again into four smaller square elements. Whereas the standard finite volume scheme is not suited to handle such a non-conforming adaptive mesh, the SUSHI scheme is compatible with these non-conforming volume elements.

We introduce the relative error in $L^2(Q_T)$ between the exact and the numerical solution as well as the experimental order of convergence

$$err(u) = \frac{\|(u_{exact}(\mathbf{x},t_n) - u_{D,\delta t}(\mathbf{x},t_n))\|_{L^2(Q_T)}}{\|(u_{exact}(\mathbf{x},t_n))\|_{L^2(Q_T)}}, \quad eoc = \frac{log(err(u_i)/err(u_{i+1}))}{log(h_{D_i}/h_{D_{i+1}})},$$

where u_i is the solution corresponding to the space discretization D_i. Table 1 shows the error using a uniform square mesh with various mesh sizes and time steps in the four first lines. Note that the scheme is only first order accurate with respect to time; therefore in order to obtain second order convergence we choose δt proportional to h_D^2. We also compare the error for the approximate saturation using a uniform mesh and an adaptive mesh with a similar number of unknowns. In both cases: about 300 unknowns (line 2 - line 5) and 1200 unknowns (line 3 - line 6), the adaptive mesh compared to the fixed one provides slightly better results for the saturation $c(u)$. The observed computational gain is rather small (about $10-20\%$), which is due to the fact that the area of high gradients of c is comparatively large.

Mesh	N	h_D	N_{unk}	$err(u)$	$err(c(u))$	$eoc(u)$
Uniform	25	0.2	85	$2.40 \cdot 10^{-2}$	$1.60 \cdot 10^{-5}$	-
Uniform	100	0.1	320	$6.09 \cdot 10^{-3}$	$4.13 \cdot 10^{-6}$	1.98
Uniform	400	0.05	1240	$1.53 \cdot 10^{-3}$	$2.90 \cdot 10^{-6}$	2.00
Uniform	1600	0.025	4880	$3.76 \cdot 10^{-3}$	$1.83 \cdot 10^{-6}$	2.02
Adaptive	200	0.143	302	$5.62 \cdot 10^{-3}$	$3.67 \cdot 10^{-6}$	-
Adaptive	800	0.071	1232	$1.32 \cdot 10^{-3}$	$2.19 \cdot 10^{-6}$	-

Table 1 Number of time steps N, mesh diameter h_D, number of unknown N_{unk}, the error of solution $err(u)$, the saturation $err(c(u))$ and the experimental order of convergence eoc.

3.2 The Haverkamp problem

We consider the case of a sand ground represented by the space domain $\Omega = (0,2) \times (0,40)$ on the time interval $[0,600]$. The parameters are given by [7]

$$\theta(p) = \begin{cases} \dfrac{\theta_s - \theta_r)}{1+|\alpha p|^\beta} + \theta_r, & \text{if } p<0, \\ \theta_s, & \text{otherwise,} \end{cases} \quad k_r(\theta(p)) = \begin{cases} \dfrac{K_s}{1+|Ap|^\gamma}, & \text{if } p<0, \\ K_s, & \text{otherwise,} \end{cases}$$

where $\theta_s = 0.287$, $\theta_r = 0.075$, $\alpha = 0.0271$ $\beta = 3.96$, $K_s = 9.44e-3, A = 0.0524$ and $\gamma = 4.74$. From θ and K, we have tabulated suitable values for the functions c and

K_c. We have taken here the initial condition $p = -61.5$, a homogeneous Neumann boundary condition for $x = 0$ and $x = 1$, the Dirichlet boundary condition $p = -61.5$ for $z = 0$ and $p = -20.7$ for $z = 40$.

We use an adaptive mesh and the time step $\delta t = 1$ to perform a test. Figure 2-(a) represents the pressure profile at various times. In this test, no analytical solution is known. Therefore we compare our numerical solution with that of Pierre Sochala [8, Fig. 2.6, p. 35] which is obtained by means of a finite element method. Our results are quite similar to his. Figure 2-(b) shows the time evolution of the mesh at different times corresponding to the pressure profiles in Figure 2-(a).

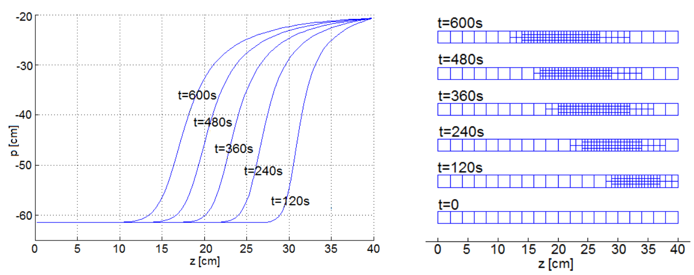

Fig. 2 Time evolution of the pressure p and the adaptive mesh.

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